Validation of Output from an Agent-Based, Double-Auction Market (7/6/2011 11:27 PM)

Introduction

There are many different models in the literature (LeBaron, 2000 provides many examples) which attempt to simulate financial markets accurately and effectively. Agent-based modeling in particular is widely used for the modeling of financial markets, specializing in systems with interacting, heterogeneous agents that are difficult to analyze using purely analytical techniques. Financial markets are a well-studied area that commonly uses agent-based techniques.

This study is an assessment of volatility clustering in a simulated price stream based on a model that differs from most of the current literature in two major ways. First, the agents simulated are differentiated by investment horizons, risk tolerance, and budget constraint levels. No switching models are used, nor are any assumptions about fundamental price levels used in the trader models. Second, the exchange mechanism is that of a simulated electronic double-auction market, which is a change from the usual market-maker approach to the exchange mechanism.

For this study, we assessed three different models of trader portfolio decision making. The first is a model of rational traders optimizing the certainty equivalent of their next-period portfolio. The next two are regret models where a regret term is added to the certainty equivalent term. The Regret and Pride models take into account the feelings of “winning” or “losing” that an investor is likely to feel after making a transaction.
The first type of regret model used in this study is the certainty-equivalent-based regret formulation. For a market order, the model calculates regret and pride as the difference between the desired and optimal allocation of wealth. For a limit order, we are interested in whether the price after buying or selling is expected to increase or decrease. The difference in the actual buy/sell price and what the market is expected to do after the transaction is used to calculate the certainty-equivalent in the case of a limit order.

The second type of regret model used in this study is price-based regret. In the case of a sell transaction, the trader will experience regret/pride if the share price received is less than/greater than the price for the asset. In this model, we use only the last price paid and do not consider any share accounting or price averaging. In the case of a buy transaction, the metric by which regret or pride are measured is the expected next-period market price. The price difference between the buy/sell price and the expected price is used directly to represent pride and regret rather than being used to calculate a certainty equivalent.

Data are often observed daily at the open or close of the trading day over the course of several years, allowing for the sample size to be large enough for statistical calculations that are meaningful. Indeed, the ARCH model used in this study for verification was designed to compensate for the inherent problems in time series data of this kind. Another data type which is collected is at the rate of transaction, called ‘tick’ data. The agent-based simulation used in this study is one which attempts to model this potentially difficult level of data frequency.

Persistent, clustered volatility is a well-known characteristic of financial time series data. In this study, we use several ARCH/GARCH models to assess the presence and persistence of volatility clusters in the simulated data. The purpose is validation of the combined regret/double-auction electronic market
model. If there are indeed persistent volatility clusters in the simulated data we analyze, then we will be able to make an argument that the model has been validated with regard to this measure, since such volatility clustering is considered to be a stylized fact of financial markets.

**Past Work and the Uniqueness of This Model**

A great deal of work has been done in the areas of agent-based modeling of financial markets and data validation of such models. One of the pioneers in the area of computational intelligence and its use in financial markets is Nobel Laureate Herbert Simon (Chen, 2005). Simon was a cognitive psychologist and computer scientist specializing in decision theory and artificial intelligence. Simon applied these studies to economics to form models that are not based on the assumptions of traditional economics but instead on the actual decision processes that people use. In this way he was able to make computer simulations that are far more accurate in simulating real data. This simple idea of using human decision practices in computer simulations changed the way many people look at economics and was the starting point of modern agent-based modeling. Since Simon’s groundbreaking ideas, there have been many models implementing the ideas of human decision-making. With these different models, one needs to be able to test the validity of the data being simulated.

Some characteristics that have become well-known as being common (that is, stylized facts) in real financial data are significant numbers of outliers resulting in ‘fat-tails’, volatility clustering, and non-linearity. There are various methods used to test for these phenomena including the Kaplan test, the BDS test, and ARCH/GARCH modeling (Chen, 2001). The Kaplan and BDS tests are quite good at showing the presence of linearity and dependence. However, the ARCH/GARCH modeling method is far more robust and more commonly used for measuring all of these phenomena and more (Bahadur, 2008; Marchisi, 2000; Engle et. al, 2007).
Many agent-based models of trader behavior have been advanced in the last 20 years (Chiarella, 2009 provides a review of these models). Most, such as are found in Chiarella and Iori (2002), Chiarella, Iori, and Periello (2009), and others, depend on the notion of two classes of traders: so-called chartists, who respond and speculate on price changes, and so-called fundamental traders, who speculate with regard to an underlying, fundamental price known only to them. Others (Hommes 2005, Vigfusson, 1996) have been concerned with single-agent switching behavior between the two classes.

One contribution of this work is the introduction of the rational, certainty-equivalent maximization model in the context of the portfolio-balancing decision to submit a market or limit order, or to do nothing. This is a departure from the current literature in that our model allows for heterogeneity in trader risk tolerances. When we add the regret components to this rational model, the simulated traders in those situations cannot be described as rational. However, the basis for the regret model is the rational, certainty-equivalent maximization model, which is briefly presented below.

In order to implement this model, models for trader assessment of the probability of trade execution in the next trading period for both market and limit orders are necessary. This is not the focus of this paper, though the models and a brief discussion are presented as Appendix II.

A second contribution of this work concerns the simulated electronic market for the single asset model (along with a second investment alternative, which, in this work, is cash) traded by our simulated investors. We present price time series from a simulated, electronic double auction market. This market does not feature a market maker, which is a major departure from current standard practice in the literature. Our model is similar to other double auction models (Rosu, 2009; Anufriev and Panchenko, 2009), however we allow for any share volume level over 100 (Rosu’s analytical approach features traders trading a single share) and orders may remain on the book for a period of time.
(measured in ticks) determined by the modeler (Aufriev and Panchenko do not allow orders to remain on the book in their model).

**The Electronic Double-Auction Model – a Summary**

The electronic double-auction market model used in this study was designed and generated at New Mexico Tech. The market features a single asset that is traded by 50 simulated traders over at least 12,889 trading events. In this model, a trading event is either the filling of a trader order or the filling of all possible orders that could fill the most-recent trader order. Traders have the option of keeping all or part of their budgets in cash; there is no risk-free asset available in the models analyzed for this study.

The discrete process by which prices are generated is as follows:

1. Traders develop trading strategies for their respective current investment horizon
2. Orders are submitted to an order list
3. Orders are randomly selected from the list; when an order is selected, either a transaction occurs (see below) or the order is added to the limit-order book or the appropriate market-order queue.
4. If either of these conditions are met by the incoming order, a simulated transaction occurs:
   a. The order is a limit order that is greater than the current (best) ask or less than the current (best) bid
   b. The order is a market order and there are either market orders or limit orders on the opposite side of the book
5. If a transaction occurs, trader budgets and volumes are adjusted, all filled orders are removed from the book, the transaction price(s) is/are posted, and the process begins again with step #1.
6. If no transaction occurs, the next order is randomly selected, which is represented by step #3.

This model is written in VBA for Excel, and runs in later versions of Excel. Individual simulations generating data such as those analyzed as part of this work require (usually) from 2 to 4 hours to run.

Double-auction model features that the modeler may alter include:

- Trader risk tolerances – these are based on a CARA utility function
- The number of short shares allowed
- The number of ticks before a short position must be liquidated
- The number of ticks that a market or limit order may remain in the queue/on the book
- The number of no-transaction time periods before a designated trader initiates a transaction, and the protocol of this liquidity-boosting transaction
- The budget constraint for each trader
- The historical data analysis horizon – the number of ticks that the trader will utilize in assessment of the mean and standard deviation of the individual return distributions
- The investment horizon – the number of ticks that represents the trading horizon for each trader

In the current model, the data-analysis and investment-horizon models are identical for all traders. Once a trader reaches the end of an investment horizon, new probability information and new certainty equivalents are computed. The trading strategy is then determined for the period, and this strategy will be the basis for submitted orders until either the trader has successfully traded or the period comes to an end and the process begins again.
Further details about the model are available upon request, and a flowchart of the electronic double-auction model is attached to this document as Appendix I.

**ARCH/GARCH**

The Autoregressive Conditional Heteroskedasticity (ARCH) model was developed by Robert Engle (1982) and then generalized by Tim Bollerslev (1986). ARCH and Generalized ARCH, or GARCH, allow for a model in which the volatility changes over time, as financial market time series price data often do. In financial data, such as that from a stock exchange, volatility tends to cluster; that is, low volatility tends to follow low volatility and high volatility tends to follow high volatility (Engle, 1982). This is a reoccurring trend in exchange data that is taken on a daily basis (Bahadur, 2008) and even appears when intra-day data is considered, as will be seen later. Until the advent of the ARCH model, no econometric models took this volatility clustering into account.

Testing to see whether an ARCH or GARCH model is a good for the data can be a good way to check for volatility clustering and non-linearity (Bahadur, 2008; Engle et al., 2007; Chen, 2005). This becomes obvious when one considers the fact that ARCH and GARCH models are used to model systems where this is a defining trait. Because of this, many software packages include tests to determine the validity of using an ARCH or GARCH model on the data. The results for these tests can be used to determine whether the model is accurate for the data as well as the presence of ARCH/GARCH effects and volatility clustering in the data. The latter is what will be utilized in this paper for the validation of simulated data.
GARCH modeling and testing is often done using data collected at the daily frequency and at the transaction frequency, though the latter is not as common. Even so, it has been proposed that this so-called ultra-high frequency data can be modeled using GARCH (Engle et al., 2007).

**The Model**

The model is an agent-based model, utilizing 50 agents who aim to optimize their next-period portfolio certainty equivalent. Each trader has a specific budget and risk aversion measure. Should a trader ever go broke, that trader is replaced by another trader with the same starting conditions and budget as the original trader. The traders will either choose to submit a market order, submit a limit order, or do nothing. Each trader has an investment amount $V_{pi_t}$ at time $t$ to allocate to risky or riskless assets.

Traders form a return probability distribution over their respective investment horizons $f_k(R|T_k) \forall i = 1, ..., I$ which is used in the calculation of the expected utility in terms of trader k’s investment horizon $t \rightarrow T$:

$$EU(W_{kT}) = \int_{-\infty}^{\infty} u_k(R|T)f_k(R|T)dR$$  \hspace{1cm} (1)$$

where $u$ denotes utility and $u_k(R|T)$ is a constant absolute risk aversion utility function over the range of $R$. The returns for this formulation are assumed to be normal. Of note is that traders can and do have different investment horizons, which results in heterogeneous trading strategies. The certainty equivalent (CE) of the expected utility, $\hat{W}_i$ for all k traders is maximized along with the CE so to maximize a trader’s CE, we can simply maximize the trader’s expected utility. Maximization of the CE is equivalent to maximization of the expected utility. In this formulation, which is based on the work of Lintner (1965), the certainty equivalent depends on the amount invested. Thus, maximization of the certainty equivalent corresponds to selection of the optimal investment amount, or portfolio weight, for each
asset under consideration. Each trader invests his wealth $V_{pk}$ among the various risky and riskless assets at any time $t$:

$$\max_{V_{ik}} \sum_{k=1}^{n}[V_{ik}(1 + ER_{ik}) - 0.5\theta_k^{-1}\sum_{j \neq i} V_{ik}V_{ij}\rho_{ij}\sigma_j](1 + r_f)^{-T} + V_{fk} \quad (2)$$

$$\text{st } \sum_{k=1}^{K} V_{ik} + V_{fk} = V_{pk} \quad (3)$$

Again, $V$ is the currency amount to be allocated among assets while $ER, \sigma,$ and $\rho$ are the expected return, standard deviation, and correlation measures for all $I$ assets and $\theta$ is the risk aversion measure of the investor. This is the general form of the model and the CE of each asset has its own, simplified version.

**No Regret**

For the model not utilizing any form of regret, the model above can be restated as:

$$\hat{W}_{kt}^0 = [V_{kt}^0(1 + ER_{kt}) - 0.5\theta_k^{-1}(V_{kt}^0)^2\sigma_k^2](1 + r_f)^{-1} + (V_{pkt} - V_{kt}^0) \quad (4)$$

The term on the far right reflects the unallocated funds from the initial budget constraint. That is, it represents the CE for doing nothing and instead keeping it as cash to be invested at next trading period.

**Certainty Equivalent-Based Regret**

The certainty equivalents for the CE-based model are broken into 2 different types: that of the market order and that of the limit order. A market order is one which a certain number of shares is bought or sold at the current market price. A limit order is one in which the trader submits a price in an attempt to buy or sell an asset. Each of these order types can once again be broken down into 2 types: purchase, or ‘bid’, and sell, or ‘ask’. For the market orders, we are interested in the difference between the optimal
allocation, $V_{kt}^\ast$, and the desired allocation, $V_{kt}^0$. We define this difference for both market and limit orders as:

$$\delta_{kt} = V_{kt}^\ast - V_{kt}^0 \quad (5)$$

For a market purchase order, we calculate the certainty equivalent as:

$$\hat{\mathcal{W}}_{kt}^M = \hat{\mathcal{W}}_{kt}^0 + \left[ \delta_{kt} \left( 1 + E R_{kt} \right) - 0.5 \theta_k^{-1} \sigma_{kt}^2 \right] \left( 1 + \eta_f \right)^{-1} + (V_{pkt} - V_{kt}^0 - \delta_{kt}) \quad (6)$$

and the market sell order certainty equivalent as:

$$\hat{\mathcal{W}}_{kt}^M = \hat{\mathcal{W}}_{kt}^0 - \delta_{kt} E[P_t] + (V_{pkt} - V_{kt}^0 + \delta_{kt}) \quad (7)$$

where $E[P_t] = P_{t-1} (1 + E R_{kt})$ is the expected price for the upcoming trading period. The expected certainty equivalent can then be calculated using the probability of trade execution, $q$, and the previously calculated no-regret and market-order certainty equivalents:

$$E[\hat{\mathcal{W}}_{kt}^M] = q_{kt} \hat{\mathcal{W}}_{kt}^M + (1 - q_{kt}) \hat{\mathcal{W}}_{kt}^0 \quad (8)$$

The number of shares to be purchased is simply the difference between our optimal and desired allocation amounts divided by the current asset price:

$$S_{kt} = \frac{|\delta_{kt}|}{P_{t-1}} \quad (9)$$

Limit orders differ from market orders in that they are driven by the preference of traders to buy low and sell high and the ability to submit the price a trader wishes to trade at. However, the probability of executing a trade will depend greatly on what that price is and will decrease as buyers’ prices go down and sellers’ prices go up. A uniform probability distribution is assumed for the range of prices a trader will consider so to find the maximum certainty equivalent we must determine the upper and lower
bounds of this range. Once this maximum expected certainty equivalent is found, we can compare it to the other certainty equivalents and choose the best trading strategy. Once the optimal bid price, $B_{kt}$, is found, we then use it to calculate the actual certainty equivalent of the successful purchase order:

$$\hat{W}_{kt}^{L} = \hat{W}_{kt}^{0} + \left[ \delta_{kt} \left( 1 + \frac{\mathbb{E}[p_{kt}]}{B_{kt}} \right) - 0.5 \theta_{kt}^{-1} \sigma_{kt}^{2} \left( 1 + r_{f} \right)^{-1} \right] \left( V_{p_{kt}} - V_{kt}^{0} - \delta_{kt} \right) \quad (10)$$

where

$$1 + \frac{\mathbb{E}[p_{kt}]}{B_{kt}} = \frac{1 + E_{R_{kt}}|B_{kt}}{B_{kt}} \quad (11)$$

so that, if the bid price is expected to increase in the next trading period, the CE also increases relative to the market order CE. For limit sell orders, we find the certainty equivalent as follows:

$$\hat{W}_{kt}^{L} = \hat{W}_{kt}^{0} - \delta_{kt} A_{kt} + \left( V_{p_{kt}} - V_{kt}^{0} + \delta_{kt} \right) \quad (12)$$

where $A_{kt}$ is the ask price submitted by trader $k$ at time $t$. As before, we calculate the expected certainty equivalent using the probability and the limit order and no regret certainty equivalents:

$$E[\hat{W}_{kt}^{L}|q_{kt}^{L}] = q_{kt}^{L} \hat{W}_{kt}^{L} + (1 - q_{kt}^{L})\hat{W}_{kt}^{0} \quad (13)$$

The number of shares to be purchased by a bid order is calculated as:

$$S_{kt} = \frac{\delta_{kt}}{B_{kt}} \quad (14)$$

and similarly, for an ask order:

$$S_{kt} = \frac{|\delta_{kt}|}{A_{kt}} \quad (15)$$
Of course, just because a trader attempts to buy or sell does not mean that the transaction is guaranteed to happen. If the order is left unfilled, the certainty equivalent is the difference between the targeted certainty equivalent and the larger value of the other two. For a market order:

\[ \hat{W}_{kt}^M - \max(\hat{W}_{kt}^L, \hat{W}_{kt}^0) \]  

(16)

and for a limit order:

\[ \hat{W}_{kt}^L - \max(\hat{W}_{kt}^M, \hat{W}_{kt}^0) \]  

(17)

**Price-Based Regret**

Price based regret is the difference between the current price and what might have been. It is calculated by taking the difference between the last share price for the asset \( P_{k,t-r} \) and the price at the time of sale \( P_{k,t} \) times the number of shares transacted \( S_t \). For a seller, we would model it as

\( S_t(P_{k,t-r} - P_t) \) and for a buyer as \( S_t(P_t - P_{k,t-r}) \). Thus, if \( P_t < P_{k,t-r} \), regret is felt by the buyer but pride is felt by the seller. Once calculated, this number is added to the no-regret certainty equivalent to get the price based regret certainty equivalent.

**Hypotheses and Testing Methods**

The purpose of this study is to test the limit order success probability distribution using an agent-based double-auction market model with no fundamental investors and two of which also implement pride and regret. The first pride and regret model is one which utilizes the CE-based regret model as described above while the other makes use of the price-based regret model. Each of these hypotheses
can be broken down into the 2 smaller hypotheses: that there exists both volatility clustering and non-linearity in each of the models. Stated formally:

$H_{1a}$: Volatility clustering exists in the simulated, non-regret data

$H_{1b}$: Non-linearity is present in the simulated, non-regret data

$H_{2a}$: Volatility clustering exists in the simulated, CE-based regret data

$H_{2b}$: Non-linearity is present in the simulated, CE-based regret data

$H_{3a}$: Volatility clustering exists in the simulated, price-based regret data

$H_{3b}$: Non-linearity is present in the simulated, price-based regret data

The null hypotheses for these are simply that there is no evidence of non-linearity or volatility clustering, or:

$H_{0a}$: There is no evidence of volatility clustering in the data

$H_{0b}$: There is no evidence of non-linearity in the data

To test this, we first take real data and run the GARCH modeling process built in to the econometric software EViews and note the ARCH/GARCH characteristics. We then use the agent-based model described above to simulate data. The transaction prices corresponding to the simulated data are recorded in a spreadsheet for analysis. These transaction prices represent tick data that one would find in a real stock market and therefore are tested for ARCH/GARCH effects in the same way that one would test real data. In this way, we can validate the simulated data and the model’s usefulness in simulating real markets.
The GARCH model chosen to validate most of the data used in this study was a GARCH(1,1) model. This model is the most commonly used for data validation of this kind. There were a few data sets that were unstable using the standard GARCH(1,1) and so slight modifications to the model were made. For two of the data sets an EGARCH(1,1) was used, for 3 data set the GARCH(1,1) model was used but a student’s t distribution was assumed for the error term rather than the standard normal distribution, for 2 sets a GARCH(1,2) model was used, and for 8 data sets the IGARCH(1,1) was used. The EGARCH model is one in which the shock factor is broken into its constituent positive and negative parts, which allows for analysis of the positive and negative shocks individually. Although this aspect is unnecessary for this study it allows for stability of the model in some cases. The student’s t distribution is acceptable when the data are particularly leptokurtic, which is the case in the vast majority of financial data. The GARCH(1,2) model is acceptable when the residuals have ARCH characteristics, which ours do. The IGARCH model is similar to the GARCH(1,1) model with the restriction that the persistence is always 1. This model is recommended any time models are unstable but, because the persistence is a point of interest, it was only used when other models did not work. The residuals from the GARCH(1,1) are still somewhat leptokurtic but choosing the best model for each data set is beyond the scope of this study. However, it is necessary that the models be stable and that the results from the modeling make sense, which is why the previously stated changes were made. What is of importance is that the simulated data show significant signs of ARCH and GARCH characteristics.

One small dataset was selected for this study for comparison purposes; Euro-Dollar exchange data from March 29th, 2011. Over one hundred twenty thousand (123,775) data points were used from this date, all ask prices regarding US Dollar to Euro exchange. The simulated data used in this study include 18 sets taken from the agent-based model without the regret/pride model, 11 sets using the CE-based regret model and 18 using the price-based regret model. The simulated sets include approximately
13,000 to 28,000 data points. The fact that the number of points in the simulated data sets is an order of magnitude lower than that of the real data is a limitation of the software used to generate the data and could not be helped. However, given the weighting of old data inherent in the models, the differences in the analysis will be negligible if they exist at all.

**Results**

Table 1 (below) shows the results from the GARCH modeling software with regard to the real data, 18 sets of the simulated data without regret, 11 sets of data with CE-based regret, and 18 sets utilizing price-based regret. The Euro-Dollar exchange data set is labeled as ‘Real’, the simulated data sets without regret as ‘NM’ and ‘NJM’ followed by a number, the simulated data sets with CE-based regret are labeled as ‘CEM’ followed by a number, and those sets with price-based regret as ‘PM’ followed by a number. Some of the data sets’ names have an (E), (I), (t), or (12) after them. These represent sets which required an EGARCH, IGARCH, student’s t distribution, or GARCH(1,2) model, respectively. The ‘Persistence’ is the sum of $\alpha_1 + \beta_1 (+ \beta_2)$ and represents the effect a change, or ‘shock’, in the data has on future values within the set. The ‘Lags’ column represents the number of lags required before the set shows signs of clustering with a confidence of 95% or better. That is, it represents the number of lags required before we can reject the null hypothesis that there is no clustering within the data with 95% confidence.
Looking at the real data set, we can see that a high persistence and low number of lags required for 95% confidence in the Q statistic is what we would expect in financial data. Below is a graphical representation of the persistence data (Figure 1).
Though not shown, of note is that the p-values for $\alpha_1$ and $\beta_1$ (and $\beta_2$) are nearly all < .001, meaning that we can be 99.9% sure the values for $\alpha_1$ and $\beta_1$ are not 0. Persistence of above 0.6 is considered high and above 0.9 is considered to be very high. For 3 of the sets in the non-regret group, models other than the standard GARCH(1,1) were required. In this group, we see many examples of high persistence but only 3 of very high persistence like we would expect from real financial data. In the CE-based regret data, however, we see many samples of very high persistence along with only 3 high persistence sets.

Unfortunately, we also see an example of persistence that is quite low at 0.24 in the data set CEM3.4. Only 2 sets from the CE-based group needed any model besides the GARCH(1,1). The price-based regret group had many examples (56%) of sets that could not be modeled using GARCH(1,1) and half of those unstable sets required an IGARCH(1,1) model to force them into stability. However, once they were properly modeled they showed high to very high persistence.

In the simulated non-regret data, we mostly see a number of lags ranging from 1 to 4, which are very low and good indicators of volatility clustering. There is, however, one anomalous set which never
reaches the 95% certainty level. In the CE-based regret data sets, we again see low numbers of lags required, ranging from 1 to 8. Although this is more lags than required by the non-regret data, it is still well within the acceptable number of lags. In this group of data we also see 2 more sets which never reach the 95% confidence level. In the price-based regret group, we again see good evidence for volatility clustering with lags ranging from 1 to 10, and again we see 2 that never reach the 95% confidence level. More important and relevant to this study is the fact that all of the data show these characteristics, indicating that they are similar to each other.

When comparing the data sets to each other there is very little difference between the no regret, CE-based and price-based regret groups. The average persistence in these groups was 0.81, 0.87, and 0.89, respectively. When we compare the number of Q-statistic lags, we find that these, too, are quite similar. The average number of lags required for the no regret and CE-based regret are 1.65 and 1.78. The price-based regret, however, is 3.38, making it nearly twice that of the other two. Determining whether or not this is significant is difficult. For now, we conclude that the difference is not significant because all models have a very low value and are within the acceptable number of lags.

To check for differences that may occur between computers running the data, below is the same data ordered by the computer it was generated on (Table 2). The reason for doing this is to test whether different computers generate significantly different data because of subtle differences in the software versions used, differences in processor speeds, or some other minor differences between the computers. Any significant differences in the data across computers needs to be accounted for and could be subject of future research.
# Table 2: Data by Computer

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<th>Computer 2</th>
<th></th>
<th>Computer 3</th>
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<td>Lags</td>
<td>Data Set</td>
<td>Persistence</td>
<td>Lags</td>
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<td>1</td>
<td>PM2.1 (12)</td>
<td>0.878262</td>
<td>10</td>
<td>PM3.1</td>
</tr>
<tr>
<td>PM1.6</td>
<td>0.999384</td>
<td>8</td>
<td>PM2.2</td>
<td>0.762152</td>
<td>1</td>
<td>PM3.2 (t)</td>
</tr>
<tr>
<td>PM1.7</td>
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<td>1</td>
<td>PM2.3</td>
<td>0.997858</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>PM1.8 (t)</td>
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<td>6</td>
<td>PM2.4</td>
<td>0.993144</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>PM1.9</td>
<td>0.965493</td>
<td>4</td>
<td>PM2.5 (I)</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>PM1.10 (t)</td>
<td>0.919791</td>
<td>∞</td>
<td>PM2.6</td>
<td>0.784109</td>
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<td>NJM1.1 (12)</td>
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<td>1</td>
<td></td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

A (E) indicates EGARCH was used, (t) indicates a student’s t distribution was used, (12) indicates a GARCH(1,2) was used, and (I) indicates IGARCH was used.

The average persistence for computer 1, 2, and 3 are 0.86, 0.88, and 0.82, respectively. The average number of lags required for these same computers is 2.28, 2.36, and 2.40. No distinct differences are immediately apparent in the sets generated. Shown below in Figure 2 is a graphical representation of the persistence data of Table 2.
Although each of the computers’ data sets does not show distinct differences when all models are accounted for, it is important to note the difficulty in modeling price-based regret data generated by Computer 1 (see Table 3, below).
Table 3: Price-based Regret on Computer 1

<table>
<thead>
<tr>
<th>Data Set</th>
<th>Persistence</th>
<th>Q-stat lags</th>
</tr>
</thead>
<tbody>
<tr>
<td>PM1.1 (E)</td>
<td>0.806641</td>
<td>2</td>
</tr>
<tr>
<td>PM1.2 (I)</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>PM1.3 (I)</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>PM1.4 (I)</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>PM1.5 (I)</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>PM1.6</td>
<td>0.999384</td>
<td>8</td>
</tr>
<tr>
<td>PM1.7</td>
<td>0.626677</td>
<td>1</td>
</tr>
<tr>
<td>PM1.8 (t)</td>
<td>0.742831</td>
<td>6</td>
</tr>
<tr>
<td>PM1.9</td>
<td>0.969493</td>
<td>4</td>
</tr>
<tr>
<td>PM1.10 (t)</td>
<td>0.919791</td>
<td>∞</td>
</tr>
</tbody>
</table>

A (E) indicates EGARCH was used, (t) indicates a student’s t distribution was used, (12) indicates a GARCH(1,2) was used, and (I) indicates IGARCH was used.

Of the ten data sets generated in this way, seven are unstable using the standard GARCH(1,1) and four of those unstable models require the IGARCH(1,1) model, which forces stability, in order to become stable. The persistence for this group is 0.91 and the average number of lags is 2.78. These numbers are slightly higher than the larger groups of data but not startlingly so.

Table 4 (below) shows a summary of the results from this study arranged by model. Note the lowest persistence for the CE-based regret; the number in parentheses represents the lowest number if the outlier is removed. Also note that the maximum number of lags only accounts for the models which converge and therefore have non-infinite values.

Table 4: Results summary

<table>
<thead>
<tr>
<th></th>
<th>Lowest Persistence</th>
<th>Average Persistence</th>
<th>Max Lags</th>
<th>Average Lags</th>
</tr>
</thead>
<tbody>
<tr>
<td>No regret</td>
<td>0.61</td>
<td>0.81</td>
<td>4</td>
<td>1.65</td>
</tr>
<tr>
<td>CE regret</td>
<td>0.24 (0.75)</td>
<td>0.87</td>
<td>8</td>
<td>1.78</td>
</tr>
<tr>
<td>Price regret</td>
<td>0.62</td>
<td>0.89</td>
<td>10</td>
<td>3.38</td>
</tr>
</tbody>
</table>
A full account of the results used in this study is available upon request.

**Discussion**

As can be seen in the results, all 3 groups of data show significant GARCH characteristics and trends and have similar persistence with only one exception. What this means is that the simulated data sets show the same non-linearity tendencies as real data. This has been seen in other agent-based models and is a common model-validation procedure in agent-based modeling. However, of interest is that the simulated data sets using the regret and pride models also show evidence of these phenomena. In fact, the regret simulations typically have a higher persistence which is common in financial data and which is not common in the non-regret data. This is important because we may now have evidence which shows that models which attempt to mimic human behavior can also be used to accurately model real world situations and, in many cases, do better than their non-regret counterparts. The one case of low persistence is likely an anomaly which is out of our control. However, even in this anomalous case, the evidence for volatility clustering is still very strong.

Volatility clustering is measured by the Q-statistic in GARCH modeling. Specifically, a low number of lags required to have a probability below 0.05 is a good indicator of volatility clustering as shown in the real data which required only 1 lag. This is common in real financial data and is what is being looked for in the simulated data. Indeed, we do find a low number of lags in the non-regret model as well as the regret models. One might assume that the larger number of lags required for some of the regret data sets to be a problem but even these larger numbers are still well within the acceptable limits. What this means is that all three of the models show strong evidence of volatility clustering.
These results mean that our null hypotheses can be rejected for all three of the models. That is, there is sufficient evidence of both volatility clustering and non-linearity in the three models. There is even evidence for the regret and pride models being more accurate in the simulation of real financial data.

In general, the differences between the models and computers used to generate the data are negligible. The data generated on computer one using the price-based regret model exhibited some interesting phenomena in that the sets generated in this way were exceptionally difficult to model. The reason for this is currently unclear but could be simply a fluke. To properly test this more data would need to be collected using this setup to get a better sample size.

**Conclusion**

Agent-based modeling is a well-developed approach for dealing with the complexity inherent in the interactions that happen between traders and market mechanisms as exchanges occur and prices are generated. These models have been shown in financial modeling to generate stylized facts present in real financial data in the areas of nonlinearity and volatility clustering. What are not widely used, however, are models simulating human behavior such as regret and pride. This study has shown that an agent-based simulation which attempts to model pride and regret can also be used to simulate real data. Additionally, the data gathered from regret simulations may be even more accurate than the data gathered from non-regret simulations. What we can be sure of is that using models which simulate human behavior can generate data with the same validity as simulations without these characteristics. We propose that these simulations should be used more often in research.

Future work should look into further use of models which implement human emotion, especially regret and pride. More research should be done in whether using human emotion could actually be more
accurate in simulating real data. Additionally, work could be done to come to a conclusion as to whether certain models work differently on different computers. It has been shown in previous work that agent-based modeling can be used to simulate financial data that is realistic and similar to real data and models utilizing human emotion are a logical next step in financial modeling.
Resources


Appendix I

1. Order Arrives
2. Order Withdrawal
   - Y: Remove Order From Book
   - N: Existing Order on Book for this Trader?
      - Y: Limit Order? (Price and...)
         - Y: Limit Ask (Sell)?
           - Y: Limit Sell Order
           - N: Limit Buy Order
         - N: Market Sell Order
      - N: Market Buy Order
3. Reject Order
Limit Buy Order

Best (Lowest) Ask?

Y

This Ask <= Best Bid?

N

Post Ask and Volume and time-step at appropriate place on the book (the ask price list); post TS

Y

Volume <= Best Bid Volume?

N

Post bid as last tick price with bid volume. Post ask price at bottom of list with adjusted volume; TS

Y

Post Bid as the Last tick price with ask volume. If Bid volume > ask volume, update bid volume on book. If bid volume = ask volume, remove bid information from book. Update records and TS for both traders

N

Update Book

** TS = Time Step
Limit Sell Order

Best (Highest)

This Bid >= Best Ask?

Volume <= Best Ask Volume?

Post Ask as the Last tick price with ask volume. If Ask volume > bid volume, update ask volume on book. If ask volume = bid volume, remove ask information from book. Update records for both traders. Update TS

Post Bid and Volume and TS** in Appropriate Place on the book (the list)

Post bid price at bottom of list with adjusted volume, TS

Update Book
Market Buy Order

Y

Sufficient Total Volume on Ask Side?

N

Sufficient Volume in One Posted Ask ?

N

Post Volume and Price from first (best) ask, subtract volume from market order and remove ask from book; repeat until either ask list is exhausted or volume request is filled; Post Trader IDs and TS for each Trade

Y

Post Ask Price as Tick Price; Post Market-Order Volume as Tick Volume; Change Volume on Posted Ask. If Volume = 0; remove Ask from book; Post all Trader IDs; TS

N

Update Book

Sufficient Volume on Book to Fill Order?

Y

Volume and Trader ID to Market Buy Order Queue; TS

N
Market Sell Order

Sufficient Total Volume on Bid Side?

Y

Sufficient Volume in One Posted Bid?

N

Post Volume and Price from first (best) bid, subtract volume from market order and remove bid from book; repeat until either bid list is exhausted or volume request is filled; Post Trader IDs and TS for each Trade

N

Update Book

Post Bid Price as Tick Price; Post Market-Order Volume as Tick Volume; Change Volume on Posted Bid. If Volume = 0; remove Bid from book; Post all Trader IDs; TS

Y

Volume and Trader ID to Market Sell Order Queue; TS

Sufficient Volume on Book to Fill Order?
Next Order is a Buy?

Limit Order?

Order Volume > Market Order Volume?

Fill Market Orders in order until volume exhausted; post ticks and TS

Post individual ticks and volumes and TS for all queued orders at limit price; post remaining volume on book at limit price

Market Sells in Queue

Add TS, Volume to Queue

Update Book
Market Sell Order Queue

Next Order is a Sell?

N

Limit Order?

Y

Limit Order?

N

Order Volume > Market Order Volume?

Y

Fill Market Orders in order until volume exhausted; post ticks and TS

N

Post individual ticks and volumes and TS for all queued orders at limit price; post remaining volume on book at limit price

Update Book

Post TS, Price, Volume on Order Book

Add TS, Volume to Queue

Market Buys in Queue

Y

Market Buy Order Queue

N
*** Stale variable is a user input indicating the number of time (steps) an order may remain on book or in queue.

On Buy and Sell Sides, Are there Orders with TS that are >= current TS + Stale Variable?***

**Y**

Eliminate them and reset Order Book

Are there Market Orders on either side queued for more TS than the Stale Variable?

**Y**

Eliminate them and reset Order Book

**N**

Order Arrives

Update Book

**N**

Order Arrives
Market Buy/Sell Order Queue Updating

New Order is Highest Volume?

Y

Post at top of Queue Book

N

Insert order into the queue with place determined by 1. Volume and, 2. Time of submission. Ties to OLDER orders – which will be listed first
Appendix II

One critical key to the formulation of rational (and, in the case of the regret models, a modified rational model) double-auction market trader behavior in a dynamic trading context are the bounds for the uniformly-distributed probability of limit-order execution. These upper and lower bounds are suggested as driven by observed prices and the current state of the limit-order book. We also propose an idiosyncratic component to the simple additive models described below.

We propose models for the computation of upper and lower bounds on the uniform probability distribution for bid success. These proposed estimates are contingent upon information available to investors from the limit book, the last observed transaction price, their investment horizons as measured in tick-lengths, and an individual factor that is contingent upon trader characteristics and condition (in terms of order and price depth on each side of the limit book) of the limit book.

The models are discussed in the context of determination of bid-price upper and lower bounds, and then in the case where determination of an optimal ask-limit price (and associated volume) is the goal. Models in the cases of both buy and sell limit orders are presented in four contexts: the case where there are both limit bid and ask orders on the book at the time of the trader’s decision, when there are bids/asks but no asks/bids, and finally in the case where there no bids or asks in view on the limit-order book.

We also present the models developed for trader assessments of the probability of execution of a market order in the next iteration of the model. This probability is not necessarily one, though because market orders are queued in the model and filled according to arrival time, the longer-term probability of a market order being filled is assumed to be one in the trader model used in this work.

Upper and Lower Bounds for the Probability of Bid Success $q_t^B | B_t$

In this case, we model the upper and lower bounds on limit-price-execution probability distributions as a function of the position of the last market price within the spread, the weighted differences between best bids and asks and other limit orders on their respective sides, and a final term reflecting the difference between the largest ask and smallest bid present on the book at the time this model is used by a given trader to impute the upper and lower bounds of the distribution for the probability of order execution within the investor’s trading interval.

Case 1: Limit Book Has Bids and Asks Posted at the Time of the Trader Decision

For the lower limit $B_{kt}$ of the bid-probability distribution for investor k at time t – the largest value for which is the minimum value for which $q_t^B | B_{kt} = 0$, and the upper limit $B_{kt}$, which is the highest value for which the investor believes that $q_t^B | B_{kt} = 1$:
\[B^L_{kt} = B_{1t} - \left| \frac{P_{t-1} - B_{1t}}{P_{t-1}} \right| - \sum_{j=1}^{l-1} e^{-(j+1)}(B_{jt} - B_{j+1,t}) + \sum_{i=1}^{l-1} e^{-(i+1)}(A_{i+1,t} - A_{it}) - h^{-1}_{kt}\sigma_{kt}P_{t-1}\]

\[B^U_{kt} = B_{1t} + \left| \frac{P_{t-1} - B_{1t}}{P_{t-1}} \right| - \sum_{j=1}^{l-1} e^{-(j+1)}(B_{jt} - B_{j+1,t}) + \sum_{i=1}^{l-1} e^{-(i+1)}(A_{i+1,t} - A_{it}) + h^{-1}_{kt}\sigma_{kt}P_{t-1}\]

\(A_{1t}\) and \(B_{1t}\) denote the lowest posted ask (of I asks) and highest posted bid (of J bids) on the limit order book at time \(t\). The last observed transaction price is \(P_{t-1}\). The investor’s tick-horizon for portfolio balancing intervals at time \(t\) is \(h_{kt}\), and the investor’s estimate of the standard deviation of returns for the current investment horizon is \(\sigma_{kt}\).

**Case 2a: Limit Book Has Only Asks Posted at the Time of the Trader Decision**

If the book condition is characterized by at least one posted ask but no posted bids, the upper and lower bounds are, in our model, estimated by

\[B^L_{kt} = P_{t-1} - \left| \frac{A_{1t} - P_{t-1}}{P_{t-1}} \right| - \sum_{i=1}^{l-1} e^{-(i+1)}(A_{i+1,t} - A_{it}) - d_{kt}P_{t-1}\]

\[B^U_{kt} = P_{t-1} + \left| \frac{A_{1t} - P_{t-1}}{P_{t-1}} \right| + \sum_{i=2}^{l} e^{-(i+1)}(A_{i+1,t} - A_{it}) + d_{kt}\]

where \(d_{kt}\) is a trader-specific adjustment factor. In the current simulation, \(d_{kt}\) is a constant with value .001 for all \(T\) trading periods.

**Case 2b: Book Has Only Bids Posted at the Time of the Trader Decision**

If the limit-order book is characterized by posted orders only on the bid side, the upper and lower bounds of the bid-success probability distribution are:

\[B^L_{kt} = B_{1t} - \left| \frac{A_{1t} - P_{t-1}}{P_{t-1}} \right| - \sum_{j=1}^{l-1} e^{-(j+1)}(B_{jt} - B_{j+1,t}) - d_{kt}P_{t-1}\]
\[ B_{kt}^U = B_{1t} + \left| \frac{A_{1t} - P_{t-1}}{P_{t-1}} \right| + \sum_{j=1}^{j-1} e^{-j} (B_{jt} - B_{j+1,t}) + d_{kt} P_{t-1} \]

Case 3: Book Has No Posted Orders at the Time of the Trader Decision

In the event that the trader is assessing bid-success probability distribution at a time when there are no orders on the book, the bounds of the distribution are formulated as a function of the trader expected return and standard deviation for the return distribution over the trader’s investment horizon:

\[
B_{kt}^L = P_{t-1} [1 + (ER_{kt} - \sigma_{kt})] \\
B_{kt}^U = P_{t-1} [1 + (ER_{kt} + \sigma_{kt})]
\]

These bounds are suggested based on some heuristic observations with regard to double-auction markets. First, if no orders appear on the ask side of the book at any time \( t \), the posted prices on the bid side may reasonably be seen to be too high, and (at least) some traders will start with a lower bound on their bid-success distributions that is higher than the best posted bid.

Similarly, if there are no orders on the book at all, this model adjusts the price model in a way that depends on the trader’s view of the underlying distribution. If the expected periodic return is exceeded by the (in the model, constant) standard deviation, there will be downward adjustment of the lower bound of the bid-success distribution.

We now turn to our model of upper and lower bounds of the probability of ask-price success in the context of computing an optimal limit order price and associated quantity.

Upper and Lower Bounds for the Probability of Ask Success \( q_{kt}^A | A_t \)

We compute the upper \( A^{U}_{kt} \) and lower \( A^{L}_{kt} \) bounds of the posited uniform ask-price execution-probability distribution such that \( A^{U}_{kt} \) is the minimum value for which \( q_{kt}^A | A^{U}_{kt} = 1 \), and \( A^{L}_{kt} \) is the maximum value for which \( q_{kt}^A | A^{L}_{kt} = 0 \). As was the case for the bid-price success distribution above, we consider the same four possible overall characterizations of the limit book at the time of the trader decision.

Case 1: Limit Book Has Bids and Asks Posted at the Time of the Trader Decision

The limits of the ask-price success distribution are computed as a function of the last observed price relative to the best ask, the depth and price-variability of both bids and asks, and individual trader horizons and standard deviations.
Case 2a: Limit Book Has Only Asks Posted at the Time of the Trader Decision

\[ A_{kt}^l = A_{1t} - \frac{|A_{1t} - P_{t-1}|}{P_{t-1}} - \sum_{i=1}^{l-1} e^{-(i+1)} (A_{i+1,t} - A_{lt}) - d_{kt}P_{t-1} \]

\[ A_{kt}^u = A_{1t} + \frac{|A_{1t} - P_{t-1}|}{P_{t-1}} + \sum_{i=1}^{l-1} e^{-(i+1)} (A_{i+1,t} - A_{lt}) + d_{kt} \]

Case 2b: Book Has Only Bids Posted at the Time of the Trader Decision

\[ A_{kt}^l = P_{t-1} - \frac{|P_{t-1} - B_{1t}|}{P_{t-1}} - \sum_{j=1}^{l-1} e^{-(j+1)} (B_{jt} - B_{j+1,t}) - d_{kt}P_{t-1} \]

\[ A_{kt}^u = P_{t-1} + \frac{|P_{t-1} - B_{1t}|}{P_{t-1}} + \sum_{j=1}^{l-1} e^{-(j+1)} (B_{jt} - B_{j+1,t}) + d_{kt} \]

Case 3: Book Has No Posted Orders at the Time of the Trader Decision

In this case when ask-price success probability bounds are the concern, the bounds are computed in the same manner as is the case if the trader is formulating a bit.

\[ A_{kt}^l = P_{t-1}[1 + (ER_{kt} - \sigma_{kt})] \]

\[ A_{kt}^u = P_{t-1}[1 + (ER_{kt} + \sigma_{kt})] \]

The major difference between computations of limit-ask-price and limit-bid-price success distributions is in the event that the trader is formulating an ask/bid price when there are only bids/asks posted on the book. In these situations, the trader's point of reference is the last observed transaction price, and that price is adjusted according to the depth of the observed quotes on the bid/ask side of the book.
In all cases, a reference price – the best ask, bid, or the price of the most recent observable transaction – is adjusted by the price relative to the best posted ask or bid, a weighted sum of the differences between posted quotes and the best quote on either side of the book, and an idiosyncratic term that is a function of trader horizons, standard deviations, and a trader-specific factor.

These models are used in the simulations described in this paper. They have yet to be examined in an empirical context.

**Models for the Probability of Next-Iteration Execution of a Market Order**

In this model, we make the design assumption that traders are able to access information about the limit-order book, but not the market queue on either the buy or sell side. The probability of next-iteration market-order execution is the same for would-be buyers or sellers, and is based on the condition of the limit-order book and on three subjective parameters.

We define $q_{kt}^M$ as an individual trader’s assessment of the probability of execution of a market order in the next trading step of the model – this may be interpreted as the probability of execution in the very short term as formulated by the individual trader.

We define $I$ and $J$ as the number of asks and bids orders on the limit book at time $t$, with $\text{Max}(I)$ and $\text{Max}(J)$ the number of orders that the trader sees as viable – those that have a chance of being filled in the short term. We also define four parameters: $\alpha_1$, $\alpha_2$, and $\alpha_3$.

**Case 1: Bids and Asks Orders are Present on the Limit Order Book**

In this case, the probability of short-term model execution is computed as

$$q_{kt}^M = \alpha_1 + (1 - \alpha_1) \left[ \frac{.5I}{\text{Max}(I)} + \frac{.5J}{\text{Max}(J)} \right]$$

**Case 2a: Only Ask Orders Appear on the Limit Order Book**

$$q_{kt}^M = \alpha_2 + \frac{(1 - \alpha_2)I}{\text{Max}(I)}$$

**Case 2b: Only Bid Orders Appear on the Limit Order Book**

$$q_{kt}^M = \alpha_2 + \frac{(1 - \alpha_2)J}{\text{Max}(J)}$$

**Case 3: No Bid or Ask Orders Appear on the Limit Order Book**

$$q_{kt}^M = \alpha_3$$
In the simulation model discussed here, the maximum number of orders on either side of the limit-order book is 15, and $(\alpha_1, \alpha_2, \alpha_3) = (0.7, 0.8, 0.5)$.

As is the case with the limit-order-execution probabilities described above, this model has not been empirically verified. Verification of this model may be a more difficult proposition given the lack of market-queue information available from the owners of most electronic trading systems.